Calculus of Parametric Equations Answer key

(1) $x = t^4 + 1 \qquad y = t^3 + t \qquad t = -1$

$(2, -2)$

$$\frac{dy}{dx} = \frac{3t^2 + 1}{4t^3} = \frac{4}{-4} = -1$$

$$y + 2 = -1(x - 2)$$

(2) $x = \cos\theta \qquad y = \sin\theta + \cos 2\theta \qquad \theta = 0^\circ$

$= (1, 1)$

$$\frac{dy}{dx} = \frac{\cos\theta - 2\sin\theta}{-\sin\theta} = \frac{1}{0}$$

$$x = 1$$

(3) $x = e^t \qquad y = (t-1)^2 \qquad (1,1) \rightarrow t = 0$

$$\frac{dy}{dx} = \frac{2(t-1)}{e^t} = -2 \qquad y - 1 = -2(x - 1)$$

(4) $$\frac{dy}{dx} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$$

$$\frac{\frac{dy'}{dt}}{\frac{dy}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}$$

$\frac{d^2y}{dx^2} = \frac{3}{4t} > 0$ ~~[illegible]~~ concave up when $t > 0$

(5) $x = t - e^{t}$ $\quad y = t + e^{-t}$

$x = 1 - e^{t}$ $\quad y = 1 - e^{-t}$

$$\frac{dy}{dx} = \frac{1-e^{-t}}{1-e^{t}} = \frac{1 - \frac{1}{e^{t}}}{1-e^{t}} = \frac{\frac{e^{t}-1}{e^{t}}}{1-e^{t}} = \frac{e^{t}-1}{e^{t}(1-e^{t})} = \frac{-1}{e^{t}} = -e^{-t}$$

$$\boxed{\frac{dy^2}{dx} = \frac{e^{-t}}{1-e^{t}}}$$

Concave up when $\frac{e^{-t}}{1-e^{t}} > 0$

$$\frac{e^{-t}}{1-e^{t}} = 0 \Rightarrow e^{-t} = 0 \Rightarrow \frac{1}{e^{t}} = 0 \Rightarrow \frac{1}{e^{t}} > 0$$

$1 - e^{t} > 0$

$e^{t} < 1$

concave up when $t < 0$

(6) $x = 2\sin t$ $\quad y = 3\cos t$

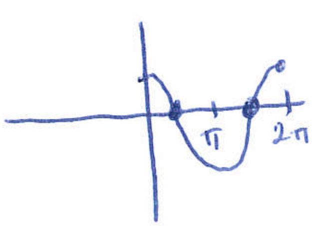
$x' = 2\cos t$ $\quad y' = -3\sin t$

$$\frac{dy}{dx} = \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = -\frac{3}{4}\sec^3 t = \frac{-3}{4\cos^3 t}$$

Concave up when $-\frac{3}{4}\sec^3 t > 0$

$\sec^3 t < 0$



$\sec t < 0$ when $\cos t < 0$

$\pi/2 \le t \le 3\pi/2$

(7) $x = 10 - t^2$ $\quad$ $y = t^3 - 12t$

$\frac{dx}{dt} = -2t$ $\quad$ $\frac{dy}{dt} = 3t^2 - 12$

Vertical

$-2t = 0$

$t = 0$

$(10, 0)$

Horizontal

$3t^2 - 12 = 0$

$t = \pm 2$

$(6, 16)(6, -16)$

(8) $x = 2\cos\theta$ $\quad$ $y = \sin 2\theta$

Vertical

$-2\sin\theta = 0$

$\theta = 0, 2\pi$

$\theta = \pi k$

Horizontal

$2\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$(2, 0)\ (-2, 0)$ $\quad$ $(\sqrt{2}, 1)$ $\quad$ $(-\sqrt{2}, -1)$

(9) $x = t - t^2 \qquad y = \frac{4}{3} t^{3/2}$

$L = \int_1^2 \sqrt{(1-2t)^2 + (2t^{1/2})^2}$

$L = \int_1^2 \sqrt{1 - 4t + 4t^2 + 4t}$

$\boxed{L = 3.168}$

$L = \int_1^2 \sqrt{1 + 4t^2}$ fnInt

(10) $x = t + \cos t \qquad y = t - \sin t$

$\boxed{L = 10.037}$

$L = \int_0^{2\pi} \sqrt{(1-\sin t)^2 + (1-\cos t)^2}$

$= \int_0^{2\pi} \sqrt{1 - 2\sin t + \sin^2 t + 1 - 2\cos t + \cos^2 t}$

$= \int_0^{2\pi} \sqrt{3 - 2\sin t - 2\cos t}$ fnInt

(11) $x = 1 + 3t^2 \qquad y = 4 + 2t^3$

$L = \int_0^{1} \sqrt{(6t)^2 + (6t^2)^2}$

$= \int_0^{1} \sqrt{36t^2 + 36t^4}$

$= \int_0^{1} \sqrt{36t^2(1+t^2)} = 6\int_0^{2\pi} t\sqrt{(1+t^2)}$

$6\left[\frac{1}{3}(1+t^2)^{3/2}\right]_0^1$

$\frac{3}{\cancel{2}}(1+t^2)\cdot\cancel{2}t$

$6\left[\frac{1}{3}(2)^{3/2} - \frac{1}{3}\right]$

$\boxed{2(2^{3/2}) - 2}$

(12) $x = \dfrac{t}{1+t}$ $y = \ln(1+t)$ $0 \le t \le 2$

$\dfrac{dx}{dt} = \dfrac{(1+t) - t}{(1+t)^2}$ $\dfrac{dy}{dt} = \dfrac{1}{1+t}$

$= \dfrac{1}{(1+t)^2}$

fnInt

$\boxed{L = 4.083}$

$L = \displaystyle\int_0^2 \sqrt{\dfrac{1}{(1+t)^4} + \dfrac{1}{(1+t)^2}}$

$= \displaystyle\int_0^2 \dfrac{1 + 1 + 2t + t^2}{(1+t)^4} = \int \dfrac{t^2 + 2t + 2}{(1+t)^4} = \cancel{\sqrt{\dfrac{(t+1)^2}{(t+1)^4}}} = \cancel{\sqrt{\dfrac{1}{(t+1)^2}}} = \cancel{\dfrac{1}{t+1}}$

$\cancel{= \ln(t+1)\Big]_0^2 = \ln 3}$

$= \cancel{\displaystyle\int \dfrac{t(t+1) + 2}{(1+t)^4}} =$

~~Let $u = t+1$~~

~~$du = dt$~~

(13) $r(t) = \langle t^2-1, t \rangle$ at $t=1$

$v(t) = \langle 2t, 1 \rangle$

$v(1) = \langle 2, 1 \rangle$

$a(t) = \langle 2, 1 \rangle \rightsquigarrow a(1) = \langle 2, 1 \rangle$

Speed $= \sqrt{2^2+1^2} = \sqrt{5}$

(14) $r(t) = \langle 2-t, 4t^{1/2} \rangle$ at $t=1$

$v(t) = \langle -1, 2t^{-1/2} \rangle$

$v(1) = \langle -1, 2 \rangle$

$a(t) = \langle 0, -t^{-3/2} \rangle$

$a(1) = \langle 0, -1 \rangle$

speed $= \sqrt{(-1)^2+(2)^2} = \sqrt{5}$

(15) $\lim_{t \to 0^+} \langle \cos t, \sin t \rangle = \langle 1, 0 \rangle$

(16) $\lim_{t \to 0^+} \left\langle \frac{e^t-1}{t}, \frac{3}{1+t} \right\rangle = \langle 1, 3 \rangle$

L Hopitals

$\frac{e^t}{1}$

(17) $r(t) = \langle 2\sin t, 2\cos t \rangle$

Total Distance $= \int_{-10}^{10} \sqrt{(2\cos t)^2 + (-2\sin t)^2} = 40$

(18) $v(t) = \langle t^2, t \rangle$ $t=0$ $(4,-1)$

$x(4) = 4 + \int_0^4 t^2$

$= 4 + \left[\frac{1}{3}t^3\right]_0^4$

$x(4) = 4 + \frac{1}{3}(4)^3$

$y(4) = -1 + \int_0^4 t$

$= -1 + \left[\frac{1}{2}t^2\right]_0^4$

$= -1 + 8$

$= 7$

$\langle 4 + \frac{64}{3}, 7 \rangle = \langle \frac{76}{3}, 7 \rangle$

(19) $v(t) = \langle \sin t, \cos t \rangle$ $t=0$ $(-3, 2)$

$x(4) = -3 + \int_0^4 \sin t$

$= -3 + \cos t\Big]_0^4$

$= -3 + (\cos 4 - \cos 0)$

$= -3 + \cos 4 - 1$

$= -4 + \cos 4$

$y(4) = 2 + \int_0^4 \cos t$

$= 2 + [\sin t]_0^4$

$= 2 + [\sin 4 - \sin 0]$

$= 2 + \sin 4$

$\langle -4 + \cos 4, 2 + \sin 4 \rangle$